Concentration of Bloch eigenstates in the presence of gauge at the semi-classical limit

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Abstract

We prove a concentration result of a Bloch eigenstate in a periodic channel under a constant gauge. In the semi-classical limit $h \to 0$ these eigenstates concentrate near a maximizer of the scalar potential of the associated Schrödinger operator, provided the constant gauge converges to a critical value from above. This is in contrast with the ground states which concentrate for *any* gauge in this limit near a *minimizer* of the scalar potential.

1 Introduction

The effect of a gauge corresponding to a null magnetic field on the spectrum of the Schrödinger operator is well known in the case of non simply connected domains [H]. In particular, the spectrum of the Schrödinger operator on the circle, manifested by the unit interval with periodic boundary conditions,

$$L_{h,P}\psi := -(hd/dx + iP)^2\psi + V\psi \; ; \; x \in [0,1] \;\; , \psi(0) = \psi(1) \; , \frac{d}{dx}\psi(0) = \frac{d}{dx}\psi(1) \; , \quad (1.1)$$

is affected by the constant gauge P. Here V is a smooth, real potential which satisfies the periodic boundary conditions as well.

Of particular interest is the effect of this gauge on the spectrum and eigenstates of (1.1) in the semi-classical limit $h \to 0$. The ground state of this operator was extensively studied (see, e.g, [K]). It is known that the normalized densities $|\psi|^2$ corresponding to the ground states converge, as $h \to 0$, to a Dirac function concentrated at the *minimizer* of V (if it is unique), while the ground eigenvalue converges to $\underline{E} := \min V$. This result is independent of the prescribed gauge P. Indeed, the ground state is invariant with respect to the shift $P \mapsto P + 2\pi h$, so the independence of the asymptotic density of the ground states is plausible. Note, however, that higher order features (such as the asymptotics of the spectral gap) are affected by the gauge [H].

On the other hand, the semi classical limits of *Bloch states* of the Schrödinger operator (1.1) are sensitive to the gauge already on the leading (macroscopic) order [E]. These are the eigenstates of $L_{h,P}$ which can be presented as $\psi = Ae^{i\theta}$ where the amplitude $A = |\psi|$ is *strictly positive* and the phase θ represents a function on the circle (i.e. $\theta(0) = \theta(1)$).

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2 Objectives

The object of this note is to study some features of the semi-classical limits of (1.1) for Bloch states. We focus on the case of *super critical Bloch states* corresponding to the eigenvalues above $\overline{E} \equiv \max(V)$. In particular, we show the existence of a sequence of Bloch eigenstates ψ_h whose amplitude $|\psi_h|$ concentrate, as $h \to 0$, near a *maximizer* of the potential x_0 (where $V(x_0) = \overline{E}$). In contrast to the ground states, the existence of such semiclassical limits are strongly conditioned on the gauge P.

Let V be a $1-$periodic potential which is maximized at a *unique* point $x_0 \in [0,1)$, such that

$$\int_0^1 (\overline{E} - V)^{-1/2} = \infty \tag{2.1}$$

where $\overline{E} := \max V$. Define

$$\underline{P} := 2^{-1/2} \int_0^1 \sqrt{\overline{E} - V} \quad .$$

The question we pose is as follows:

Is there a sequence of normalized eigenstates ψ_h of $L_{h,P}$ which, for $h \to 0$ and $P \searrow \underline{P}$, concentrate near the Dirac function δ_{x_0} while the corresponding eigenvalues converge to $\overline{E}$?

Let the function $E_0 = E_0(P) \geq \overline{E}$ defined by inverting

$$\int_0^1 \sqrt{E_0(P) - V} = \sqrt{2} P \ . \tag{2.2}$$

The main result we prove is

Theorem 1. *For any interval $[a, b] \subset]0, 2\pi[$, any sequence $\{\gamma_n\} \subset [a, b]$ and any $P > \underline{P}$ there exists a sequence $P_n \to P$ and a sequence of normalized Bloch eigenstates*

$$L_{h_n,P_n}(\phi_n) = E_0(P_n)\phi_n$$

where $h_n := \frac{\sqrt{2}P_n}{2n\pi+\gamma_n}$ such that

$$|\phi_n|^2 \to \frac{(E_0(P) - V)^{-1/2}}{\int_0^1 (E_0(P) - V)^{-1/2}} := A_0^2(P) \ . \tag{2.3}$$

holds uniformly on $[0, 1]$.

The following Corollary satisfies a partial answer to the question above:

Corollary 2.1. *There exists $P_k \searrow \underline{P}$, $E_k \searrow \overline{E}$, $h_k \to 0$ and normalized Bloch eigenstates*

$$L_{h_k,P_k}(\psi_k) = E_k\psi_k$$

such that $|\psi_k|^2 \rightharpoonup \delta_{x_0}$ as distributions.

The Corollary follows easily form Theorem 1. Fix $\gamma \in (0, 2\pi)$ and let $\hat{P}_k \searrow \underline{P}$. By Theorem 1 we can find $P_{n,k} \to \hat{P}_k$ as $n \to \infty$ and $h_n = \frac{\sqrt{2}\underline{P}}{2n\pi+\gamma}$ so that $\phi_{n,k}$ are normalized Bloch eigenfunctions of $L_{h_n,\hat{P}_{n,k}}$ subjected to the eigenvalue $E(\hat{P}_k)$ and $|\phi_{n,k}|^2 \to A_0^2(\hat{P}_k)$ uniformly as $n \to \infty$. Then, by Theorem 1 it follows

$$\lim_{k\to\infty}\lim_{n\to\infty} |\phi_{n,k}|^2 = \lim_{k\to\infty} A_0^2(\hat{P}_k) = \delta_{x_0}$$

in distributions. The Corollary then follows by taking a subsequence $\psi_n = \phi_{n,k(n)}$ where $k(n) \to \infty$ as $n \to \infty$ slow enough.

3 Proof of the main result

Each eigenstate is a critical point of the functional

$$F_h(\psi) := \int_0^1 V|\psi|^2 + |h\psi' + iP\psi|^2 \ ; \quad \int_0^1 |\psi|^2 = 1$$

Let us consider the Bloch states represented as $\psi = Ae^{i\theta/h}$ where $A > 0$ and θ satisfy the periodic condition on $[0, 1]$. Then

$$F_h(A, \theta) = \int_0^1 A^2|\theta' + P|^2 + VA^2 + h^2|A'|^2$$

Since $A > 0$ on $[0, 1]$ there is only one critical point of F_h with respect to θ, namely

$$\left((\theta' + P)A^2\right)' = 0 \quad \Rightarrow \theta' + P = \frac{\lambda}{A^2}$$

for some $\lambda \in \mathbb{R}$. Since θ satisfies the periodic boundary condition $\theta(0) = \theta(1)$ it follows

$$P = \lambda \int_0^1 A^{-2} \Rightarrow A^2(\theta' + P)^2 = \lambda^2/A^2 = P^2A^{-2}\left(\int_0^1 A^{-2}\right)^2 \ ,$$

so $\overline{F}_h(A) := \min_\theta F_h(A, \theta)$ takes the value

$$\overline{F}_h(A) = P^2\left(\int_0^1 A^{-2}\right)^{-1} + \int_0^1 VA^2 + h^2|A'|^2 \ .$$

A critical point of $\overline{F}_h$, then, satisfies

$$h^2 A'' = (V - E_h(P))A + \frac{2P^2 A^{-3}}{\left(\int_0^1 A^{-2}\right)^2} \quad , \quad A(0) = A(1), \quad A'(0) = A'(1) \ . \tag{3.1}$$

Here $E_h(P)$ is the Lagrange multiplier due to the constraint $\int_0^1 A^2 = 1$ and corresponds to an eigenvalue of the Shrodinger operator.

So, let us set $h = 0$ in (3.1) to obtain

$$A_0^2 = \frac{\sqrt{2}P}{\int_0^1 A_0^{-2}} (E_0(P) - V)^{-1/2} \ .$$

This can be solved only if $E_0(P)$ is compatible with (2.2) and, in particular, only for $P > \underline{P} := 2^{-1/2} \int_0^1 \sqrt{\overline{E} - V}$. In that case we get

$$A_0^2(P) = \frac{(E_0(P) - V)^{-1/2}}{\int_0^1 (E_0(P) - V)^{-1/2}} \ .$$

Setting

$$\lambda := \left(\int_0^1 (E_0(P) - V)^{-1/2} \right)^{-2} \quad , \quad E(\lambda) := E_0(P) \ ,$$

Theorem 1 is obtained from

Proposition 3.1. *Let*

$$\overline{\Omega}_\lambda := 2 \int_0^1 \sqrt{E(\lambda) - V} \tag{3.2}$$

and

$$A_0(\lambda) := \frac{\lambda^{1/4}}{(E(\lambda) - V)^{1/4}} \ . \tag{3.3}$$

Then, for each each $\lambda > \left(\int_0^1 (\overline{E} - V)^{-1/2}\right)^{-2}$ and $\gamma \in (0, 2\pi)$ there exists $N(\gamma) > 0$ and a solution A_h of

$$h^2 A_h'' = -(E(\lambda) - V)A_h + \lambda A_h^{-3}$$

which satisfies the periodic boundary condition on $[0,1]$ provided $h = \frac{\overline{\Omega}_\lambda}{2n\pi + \gamma}$ and any integer $n > N(\gamma)$. Moreover, $A_h \to A_0$ as $n \to \infty$.

Proof. Let $A_h = A_0 + \eta$. Then η satisfies

$$h^2 \eta'' = -\Omega_\lambda^2(x)\eta + Q(x, \eta, h)\eta + h^2 A_0''$$

where

$$\Omega_\lambda^2(x) = E(\lambda) - V + 3\lambda A_0^{-4}(\lambda) \equiv 4(E(\lambda) - V) \tag{3.4}$$

by (3.2, 3.3) and $Q(x,0,h)\equiv 0$. We now scale $x\to x/h$ $d/dx()\to \dot{()}$ and $\eta\to h\eta$ to obtain

$$\ddot{\eta}=-\Omega_h^2(hx)\eta+h\left(\hat{Q}_h(hx,\eta)\eta-A_0^{''}(hx)\right)$$

where

$$\hat{Q}_h(\eta,x):=h^{-1}Q(h\eta,x,h)=:q(x)\eta+hq_h(x,\eta)\ . \tag{3.5}$$

Let us now set

$$\eta=R\cos\Theta\ ,\quad \dot{\eta}=\Omega_\lambda(hx)R\sin\Theta\ .$$

so

$$\dot{R}\cos\Theta-\dot{\Theta}R\sin\Theta=\Omega_\lambda(hx)R\sin\Theta \tag{3.6}$$

$$\left(\dot{R}\sin\Theta+\dot{\Theta}R\cos\Theta\right)\Omega_\lambda(hx)=-\Omega_\lambda^2R\cos\Theta+hH_h(hx,R,\Theta) \tag{3.7}$$

where

$$H_h=\hat{Q}_h(hx,R\sin\Theta)R\sin\Theta-A_0^{''}(hx)-\Omega_\lambda^{'}(hx)R\sin\Theta \tag{3.8}$$

Multiply (3.6) by $\cos\Theta$, (3.7) by $\Omega_\lambda^{-1}\sin\Theta$ and sum to obtain

$$\dot{R}=h\frac{\sin\Theta}{\Omega_\lambda}H_h(hx,R,\Theta) \tag{3.9}$$

Likewise, multiply (3.6) by $-\sin\Theta/R$, (3.7) by $\cos\Theta/(R\Omega_\lambda)$ and sum to obtain

$$\dot{\Theta}=-\Omega_\lambda(hx)+\frac{h\cos\Theta}{R\Omega_\lambda}H_h(hx,R,\Theta)\ . \tag{3.10}$$

In complex notation, $Z=Re^{i\Theta}$ and (3.9, 3.10) takes the form of single equation in the complex plane $\mathbb{C}$:

$$\dot{Z}=-i\Omega_\lambda(hx)Z+ih\frac{H_h(hx,Z,Z^c)}{\Omega_\lambda(hx)} \tag{3.11}$$

where $Z^c:=Re^{-i\Theta}$. Let $\Psi_h:\mathbb{C}\to\mathbb{C}$ be the map obtained from the solutions of (3.11) at time h^{-1}, that is:

$$\Psi_h\left(Z(0)\right):=Z(h^{-1})\ .$$

We need to prove that Ψ_h has a fixed point for h sufficiently small, under the stated conditions.

Apply the transformation

$$Z\mapsto\tilde{Z}:=Z-h\frac{\cos\theta}{R\Omega_\lambda} \tag{3.12}$$

and substitute in (3.11) to obtain

$$\dot{\tilde{Z}}=-i\Omega_\lambda(hx)\tilde{Z}+O(h^2) \tag{3.13}$$

Integrating (3.13) on the interval $[0, h^{-1}]$ to obtain

$$\tilde{Z}\left(h^{-1}\right) = \tilde{Z}(0)e^{-i\overline{\Omega}_\lambda/h} + O(h)$$

where $\overline{\Omega}_\lambda = \int_0^1 \Omega_\lambda(x)dx$ as defined in (3.2).

Note that the transformation (3.12) is singular at $R = 0$, so the term $O(h^2)$ is also singular at $R = 0$. However, the conjugancy between (3.11) and (3.13) is still valid away from $R = 0$. In particular we obtain

Lemma 3.1. *The limit*

$$\lim_{h\to 0} e^{i\overline{\Omega}_\lambda/h}\Psi_h(Z) = Z$$

holds uniformly on $|Z| = r$ *for each* $r > 0$. *In particular, if* $h_n = \frac{\overline{\Omega}_\lambda}{2n\pi+\gamma}$ *then*

$$\lim_{n\to\infty} \Psi_{h_n}(Z) = e^{-i\gamma}Z \tag{3.14}$$

uniformly on $|Z| = r$.

We complete the proof by utilizing the following fixed point theorem which follows from a "soft" topological argument

Lemma 3.2. *Assume* $\Psi : \{|Z| \leq r\} \to \mathbb{C}$ *is a of continuous function. Given* $\gamma \in (0, 2\pi)$, *if* $|\Psi(Z) - e^{-i\gamma}Z|$ *is sufficiently small for* $|Z| = r$ *then* Ψ *has a fixed point* z_0 *in* $\{|Z| < r\}$.

Proof. Assume the contrary. Then the function

$$\Phi(z) = \frac{\Psi(Z) - Z}{|\Psi(Z) - Z|} : \{|Z| \leq r\} \to \{|Z| = 1\}$$

is continuous. By assumption $|\Phi(Z) - e^{i\alpha}Z|$ is uniformly small on $|Z| = r$ where $e^{i\alpha} = \frac{1-e^{-i\gamma}}{|1-e^{-i\gamma}|}$. In particular, the winding number of the mapping Ψ restricted to the circle $\{|Z| = r\}$ equals one, so the topological degree of Φ equals one as well. By degree theory [OCQ] $\Phi = 0$ must have a root in the disk $\{|Z| \leq r\}$, which is clearly impossible. □

The proof of Proposition 3.1 (and Theorem 1) now follows since $r > 0$ can be chosen arbitrary small if h is small enough. In particular, the amplitude of $\eta = A_h - A_0$ is arbitrary small for h small enough. □

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